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MATHEMATICAL MODEL OF HEAT AND MASS TRANSFER IN A BIDISPERSE POROUS MATERIAL

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A model of heat and mass processes in a body with two types of pores is considered. This model describes the initial stage of substance penetration into the porous system (or the inverse process, namely, substance extraction from the system) and takes into account convective transport in large channels. A kinetic function of impregnation (extraction) of the porous medium and the substance flux density are found for a problem with additional conditions.

Key words: *porous body, mass transfer, impregnation, two-component model.*

Introduction. A two-component transport model is usually used to describe a number of heat- and mass-transfer processes, including convective motion of the transported substance. This model can be derived from the condition of heat and mass balance in two continua with specific laws of energy and mass transfer and with a postulated law of energy and mass exchange between the components (see, e.g., [1–7]). A more generic approach is the use of methods of mechanics of continuous media for finding relations between interpenetrating continua, where the necessary closing relations are found from the corresponding problems with microscopic scales for the basic variables (see, e.g., [8–11]). A two-component (bidisperse) model of heat and mass transfer is usually used when the results predicted by a simpler one-component model differ significantly from experimental data because of an oversimplified presentation of the medium in the form of a homogeneous continuum or when the physical phenomenon has an essentially two-component character. Specific features of hydrodynamics of transport process and chemical kinetics can be often explained by using a model of a body containing pores of difference sizes (fissured-porous systems) [12, 13]. For instance, mass transfer in some media forms zones with quantitatively and qualitatively different mass exchange with the ambient medium (cavities, pockets, stagnant zones, etc.). Despite the large variety of transport phenomena (washing of sediments, desalinization of soils, heat transfer in a heterogeneous medium, filtration in bidisperse or fissured-porous media, adsorption, etc.), the models have much in common, which allows one to use the results for studying similar processes.

The solution of problems of this class is usually based on equations that describe processes occurring within sufficiently large time intervals, whereas the behavior of the heat- and mass-transfer characteristics at the initial stage (short times) has not been adequately studied. Moreover, the problem considered usually involves several dimensionless parameters (criteria) and characteristic times; therefore, the analysis is rather complicated. The initial stage of the process evolution is analyzed in the present work on the basis of a two-component model, and analytical solutions are found.

Construction of the Model and Formulation of the Problem. Construction of models of hydrodynamics and heat and mass transfer in multispecies media usually involves averaging, for instance, over the volume [8] or over the ensemble of system configurations [10]. In all variants of averaging, one has to use the procedure of closing the system of equations obtained and consider the corresponding problem at the microscale under certain assumptions about the structure of the mixture. The problem of system closing can be solved exactly [8] only by applying additional hypotheses, and this can usually be done for sufficiently rarefied systems. When simpler

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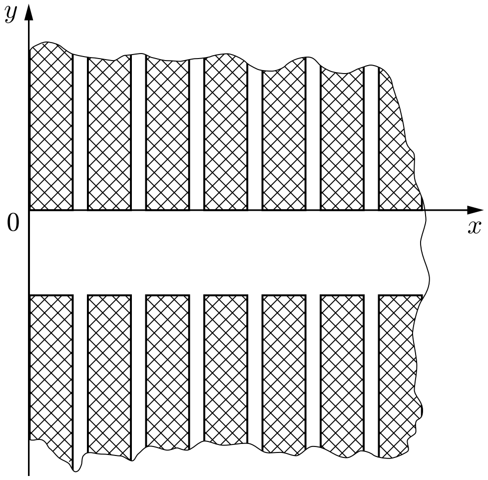

Fig. 1. Idealized configuration of pores in a semi-infinite body.

approaches are used, no averaging is performed, but a particular system configuration is chosen under certain assumptions and hypotheses about the geometry of the non-single-phase mixture and about the exchange processes between the phases (components). The use of such an approach, nevertheless, yields the same equations that are obtained by using more complicated and mathematically justified procedures associated with averaging. In the present work, we do not use averaging of microequations in deriving the constitutive equation and apply some postulates of [14] with appropriate corrections.

Let us consider an idealized system of channels in a semi-infinite body shown in Fig. 1 (borrowed from [14]). There are pores of two kinds in the half-space $x > 0$: large pores, which reach the surface $x = 0$, and small pore connected to large pores. As the heat- and mass-transfer equations in the model considered have an identical form, we further consider mass transfer. Let the concentration of the target component at the initial time ($t = 0$) be identical in all pores and equal to C_0. Then, a large pore is “opened,” and the target component starts to pass to the domain $x \leqslant 0$, where the concentration of the target component is assumed to be zero not only at $t = 0$, but also at all subsequent time instants. In contrast to [14], the model used here takes into account convective transfer of the target component in large channels with a constant velocity. The task is to find the target component flux in a large pore at $x = 0$ as a function of time. According to this presentation of the porous medium, the channels are not necessarily identical and straight. Both large and small pores can be curved. The x coordinate is the distance counted from the cross section $x = 0$ along the large channel, and the y coordinate is the distance counted from the large channel along the small pore.

The process of removal of the target component from the porous body is called extraction, and the process of substance penetration into the body is called impregnation.

Note that it is more convenient to solve another problem instead of the problem posed above. Let the target component with the concentration C_0 be present in the domain $x < 0$ at the initial time and be absent in the domain $x > 0$. When the large pore is “opened,” the process of impregnation of the porous body begins. As the transport equations are linear and contain only the derivatives of concentration with respect to the coordinate and time, we can easily find the relation between the target component fluxes in the cross section $x = 0$ during impregnation and extraction. Indeed, the concentrations of the target component in large and small pores during impregnation are related to the corresponding concentrations during extraction by simple relations: $C_j^{\text{impr}} = C_0 - C_j^{\text{extr}}$ ($j = 1, 2$). This yields the following dependence of the target component fluxes at $x = 0$ for the impregnation and extraction problems:

$$F_s^{\text{extr}} = UC_1^{\text{extr}} - D_1 \frac{\partial C_1^{\text{extr}}}{\partial x}\Big|_{x=0} = UC_0 - \Big(UC_1^{\text{impr}} - D_1 \frac{\partial C_1^{\text{impr}}}{\partial x}\Big)\Big|_{x=0} = UC_0 - F_s^{\text{impr}}. \tag{1}$$

Here, F_s^{impr} and F_s^{extr} are the target component fluxes during impregnation and extraction, respectively, and U is the velocity of convective transport of the target component; the subscripts 1 and 2 refer to the large (transport) channels and small pores, respectively. In what follows, the superscript “impr” is omitted.

The process of transportation of the target component in the large pore is described by the following equations:

$$\Big(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x} - D_1\,\frac{\partial^2}{\partial x^2}\Big)\,C_1 = -q(x,t); \tag{2}$$

$$\begin{gathered} C_1 = C_1(x,t), \qquad x \in (0,\infty), \qquad t \in (0,\infty), \\ C_1(0,t) = C_0 = \text{const}, \qquad C_1(\infty,t) < \infty, \qquad C_1(x,0) = 0. \end{gathered} \tag{3}$$

Note that the transport coefficient, at least in large pores, is not necessarily determined by molecular diffusion (see, e.g., [15]). Assuming an infinite length of the transport pore along the x axis ($l_1 \to \infty$), we obtain the relation $T_1 = l_1^2/D_1 \to \infty$.

In addition to the usual differential terms, Eq. (1) includes the term q, which describes the substance outflow from large to small pores. In the general case, the functional relation between q and other parameters of the problem depends on mass-transfer processes in the entire system of interrelated channels in the porous body and can hardly be determined exactly. Therefore, some hypotheses are used to describe mass transfer in a heterogeneous medium. In the present work, we formulate a dependence for q, which describes the process at sufficiently short times. Let us write the transport equation in the small pore with appropriate boundary conditions (see [14]):

$$\begin{gathered} \Big(\frac{\partial}{\partial t} - D_2\,\frac{\partial^2}{\partial y^2}\Big)\,C_2 = 0, \qquad y \in (0,\infty), \quad t \in (0,\infty), \\ C_2 = C_2(y,t), \qquad C_2(0,t) = C_1(x,t), \qquad C_1(\infty,t) < \infty, \qquad C_1(y,0) = 0. \end{gathered} \tag{4}$$

It is clear that the diffusion equations in the vicinity of the junction between the channels cannot be considered as homogeneous. The use of one-dimensionality is an assumption simplifying the problem formulation. Such simplifications are justified for some heterogeneous media with the use of averaging over the ensemble of system configurations, for instance, in those cases where the diffusion processes in the continuous medium are much faster than those in the disperse medium [11]. In the case considered, this occurs during mass exchange in the contact region between the large channel with the small ones.

Short Times. Short times are understood as time intervals satisfying the inequality $t \ll T_2$, where $T_2 = l_2^2/D_2$ is the characteristic time of diffusion propagation of the target component in small channels. In deriving further equations, we assume that the size of the inner (small) pore l_2 has an infinite value ($l_2 \to \infty$). Thus, the problem is considered in the approximation of a semi-infinite body with respect to both spatial coordinates. In this case, the flux q in Eq. (2) is described by the expression [14]

$$q = -4\varepsilon d^{-1} D_2\,\frac{\partial C_2}{\partial y}\Big|_{y=0} = 4\varepsilon d^{-1} D_2^{1/2} L^{1/2} C_1(x,t),$$

which follows from Eq. (4), under the additional formulated conditions. Here we use the following definition of the operator of fractional differentiation with respect to time of the half-order of the function f:

$$L^{1/2} f(t) = \frac{d^{1/2} f(t)}{dt^{1/2}} = \frac{1}{\sqrt{\pi}}\,\frac{d}{dt}\int\limits_0^t \frac{f(\tau)\,d\tau}{\sqrt{t-\tau}}.$$

As a result, the problem of determining the concentration in the porous body is reduced to one equation

$$\Big(\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x} + b\,\frac{\partial^{1/2}}{\partial t^{1/2}} - D_1\,\frac{\partial^2}{\partial x^2}\Big)\,C_1 = 0, \qquad C_1 = C_1(x,t), \tag{5}$$

where $b = 4\varepsilon d^{-1} D_2^{1/2}$, d is the diameter of the large channel, and ε is the fraction of the cross-sectional area of the small pores at their interface with the large pores, where mass transfer can occur [the interface of the large channel includes not only the areas of the entrance of the small channels, but also the solid (impermeable for the substance) part of the interface; hence, ε is the ratio of the liquid part of the area of the large pore interface to the total area of the interface, including the solid portion of the interface impermeable for the substance]. This expression yields the characteristic time $T_3 = b^{-2}$, which is the time of mass exchange between the large and small channels. Depending on the ratio between the times T_3 and T_2, certain simplifications of the problem can be used. Of greatest interest is the case with $T_2 \gg T_3$. Here, we obtain $d \ll 4\varepsilon l_2$.

Equation (5) has four terms. Depending on the situation in various space and time domains, only some of them can be essential. The case where the third term can be neglected was studied to the greatest extent (formally, $b \to 0$). When the second or fourth term can be neglected, we obtain problems that require special considerations. The problem with $D_1 \to 0$ in Eq. (5) is studied below. The main attention is paid to the case where all terms in this equation should be taken into account.

Analysis of the Problem. It seems reasonable to write Eq. (5) and additional conditions (3) in dimensionless variables:

$$\Big(\frac{\partial}{\partial\tau} + 2\,\frac{\partial}{\partial z} + 2\gamma\,\frac{\partial^{1/2}}{\partial\tau^{1/2}} - \frac{\partial^2}{\partial z^2}\Big)\,G = 0,$$

$$G(z,\tau) = C_1/C_0, \qquad z \in (0,\infty), \qquad \tau \in (0,\infty); \tag{6}$$

$$G(0,\tau) = 1, \qquad G(\infty,\tau) = 0, \qquad G(z,0) = 0. \tag{7}$$

Here $\tau = t/T_4$, $z = x/l_4$, $T_4 = 4D_1/U^2$, $l_4 = 2D_1/U$, and $\gamma = bD_1^{1/2}/U$.

Note that equations similar in form to Eqs. (5) and (6) are close to equations proposed, for instance, in [11, 13, 15, 16] and are their particular cases.

It is convenient to solve problem (6), (7) by an operation method with respect to the variable τ. The transformed problem has the form

$$\Big(p + 2\gamma p^{1/2} + 2\,\frac{d}{dz} - \frac{d^2}{dz^2}\Big)G^* = 0, \qquad G^*(0,p) = \frac{1}{p}, \qquad G^*(\infty,p) = 0, \tag{8}$$

where the asterisk indicates the Laplace-transformed quantities; p is the transformation variable. The solution of problem (8) is written in the form

$$G^*(z,p) = \exp\{z[1 - (p + 2\gamma p^{1/2} + 1)^{1/2}]\}/p, \tag{9}$$

where the main branches are chosen for the "internal" and "external" square roots; in particular, we have $\sqrt{1 + 2\sqrt{1} + 1} = 2$ for $p = 1$ and $\gamma = 1$. It seems of interest to find the target component flux on the porous body boundary, i.e., $F_s = 2G - (\partial G/\partial z)\Big|_{z=0}$. From Eq. (9), we obtain the following relation for this variable:

$$F_s^*(p) = \Big(1 + \sqrt{p + 2\gamma\sqrt{p} + 1}\,\Big)/p. \tag{10}$$

In practice, it is rather difficult to measure the instantaneous value of the flux $F_s(\tau)$ directly. Direct measurements yield only the amount of the substance that passed through the boundary $x = 0$ by the time t:

$$Q_s(\tau) = \int_0^{\tau} F_s(t)\,dt.$$

According to Eq. (10), we have

$$Q_s^*(p) = \Big(1 + \sqrt{p + 2\gamma\sqrt{p} + 1}\,\Big)/p^2. \tag{11}$$

To expand the function $Q_s^*(p)$ into a series with respect to the powers of $p^{-1/2}$, we use the formula for the generating function of the Legendre polynomials [17, 18]

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{12}$$

where $P_n(x)$ are the Legendre polynomials. There are several formulas available for calculating the values of these polynomials [17, 18]. Note that the radius of convergence of the power series with respect to t (12) depends on x for $|x| \geqslant 1$. Let us divide Eq. (11) into two terms and multiply the numerator and denominator of the second term by $\sqrt{p + 2\gamma\sqrt{p} + 1}$. Taking into account Eq. (12) and using expansion into a series with respect to the powers of $p^{-1/2}$, we obtain

$$Q_s^*(p) = \frac{1}{p^2} + \frac{p + 2\gamma\sqrt{p} + 1}{p^2\sqrt{p}} \sum_{n=0}^{\infty} \frac{(-1)^n P_n(\gamma)}{p^{n/2}}. \tag{13}$$

After that, we use the correspondence between the image p^{-n} and the original $\tau^{n-1}/\Gamma(n)$ [$n > 0$; $\Gamma(z)$ is the Euler gamma function] [17, 18]. As a result, Eq. (13) yields a solution applicable for calculations at sufficiently short times:

$$Q_s(\tau) = \tau + \sum_{k=0}^{\infty}(-1)^k P_k(\gamma)\tau^{k/2}\Big[\frac{\tau\sqrt{\tau}}{\Gamma(k/2+5/2)} + \frac{2\gamma\tau}{\Gamma(k/2+2)} + \frac{\sqrt{\tau}}{\Gamma(k/2+3/2)}\Big]. \tag{14}$$

Series (14) has an infinite radius of convergence; hence, a moderate number of its terms are necessary for calculations in the range of times of practical importance for $\gamma < 1$. The case with $\gamma > 1$ involves some difficulties in calculations because of the asymptotic behavior of the Legendre polynomials with the argument greater than unity as $k \to \infty$. Using available formulas [17, 18], we can express all gamma functions involved into series (14) via factorials of certain integers. The first term in Eq. (14) describes the convective inflow of the target component into the porous body. At short times, the third term in the square brackets of sum (14) at $k = 0$, corresponding to diffusion impregnation (without small channels), is the most essential term. The physical meaning of this term dominating at $\tau \to 0$ is the absence of the effect of the internal pores on the impregnation process at short times. The next terms in Eq. (14) introduce corrections to mass transfer, which take into account the effect of internal channels and convective transport of the target component. It should also be noted that some simple transformations of expansion (13) allow us to write the solution of the problem of determining the target component flux at the domain boundary with a specified boundary function of the general form via the fractional derivatives of this function.

From Eq. (14), we can easily find the formula for Q_s during extraction. Taking into account the relation between the variables F_s and Q_s ($dQ_s/d\tau = F_s$) and Eq. (1) in dimensionless variables, we can easily see that the first term in the expression for Q_s (14) in the case of extraction is the same as that in the case of impregnation, and there is a minus sign at the sum.

At large values of time ($t \gg T_4$, where $T_4 \ll T_2$), it is convenient to obtain an asymptotic (as $t \to \infty$) presentation of the solution, expanding Eq. (11) into the Taylor series with respect to the powers of $p^{1/2}$ and inverting the resultant expression term by term. We use the formula of the correspondence between the image p^{-n} and the original $\tau^{n-1}/\Gamma(n)$ of the Laplace transform with $n < 0$, despite the fact that this formula was justified for $n > 0$ within the framework of the classical mathematical analysis. The transformations applied yield a correct asymptotic expansion, because the search algorithm satisfies the conditions of the corresponding theorem of operational calculus [18, 19]. As a result, we obtain

$$\begin{aligned} Q_s(\tau) = 2\tau + \frac{1-\gamma^2}{2} - \frac{1}{\pi\sqrt{\tau}}\sum_{k=0}^{\infty}\frac{(-1)^k}{\tau^k}\,P_{2k+1}(\gamma)\Gamma(k-1/2)(k-1/2-\tau) \\ - \frac{2\gamma\sqrt{\tau}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\tau^k}\,P_{2k}(\gamma)\Gamma(k-1/2), \qquad \tau \to \infty. \end{aligned} \tag{15}$$

Note that the function $\Gamma(n-1/2)$ is expressed via factorials [17, 18].

Equations (14) and (15) are sufficient to obtain a complete description of the kinetic curve for impregnation. It should be noted, however, that it is reasonable to use different scales for the problem variables at high values of the parameter γ (apparently, even for $\gamma \geqslant 2$) and to use the perturbation technique [20] taking the solution of the problem of impregnation of a semi-infinite body with no convective transport as the principal approximation. The point is that the values of the Legendre polynomials $P_n(\gamma)$ in Eq. (14) at $\gamma \geqslant 2$ become fairly large at high values of n. For this reason, it is difficult to use the solution in the form of series (14) at such values of γ. Apparently, the effect of a finite number of significant digits in computer calculations is manifested here, which is similar to that described in [21] in solving the problem of a detached shock wave. Numbers exceeding the sum of the series by many orders of magnitude appear in the course of calculating the terms of the series. For a finite number of significant digits, as a result, the rounding errors are much greater than the values of the sought sum [21].

At $\gamma = 1$, inversion of formula (11) is substantially simpler:

$$Q_s(\tau) = 2[\tau + (\tau/\pi)^{1/2}]. \tag{16}$$

It follows from Eq. (16) that convective motion leads to a linear increase in the target component mass as $\tau \to \infty$. In accordance with the discussion above, diffusion impregnation (with no allowance made for small channels) prevails at short times.

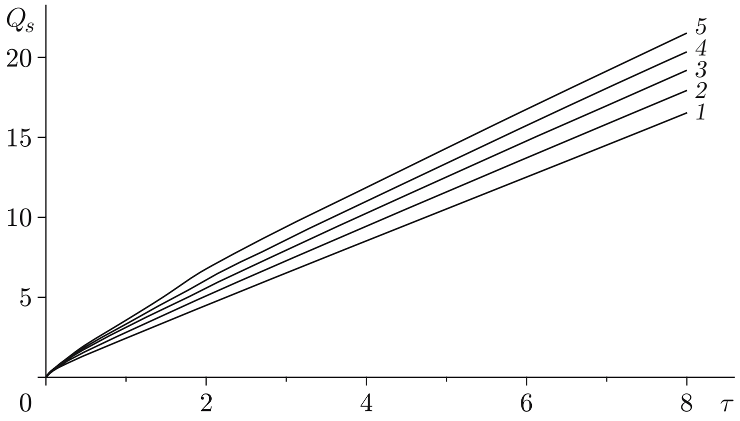

Fig. 2. Dependence $Q_s(\tau)$ for $\gamma = 0.01$ (1) 0.5 (2), 1.0 (3), 1.5 (4), and 2.0 (5).

Figure 2 shows the dependence $Q_s(\tau)$ for different values of the parameter γ. Dependence (16) was used to check the calculations by Eqs. (14) and (15).

It follows from Eq. (14) that it is possible to obtain a single series in terms of the powers of τ, whose terms are some other polynomials, which are linear combinations of the Legendre polynomials. Let us write Eq. (14) in a more compact form.

Integrating Eq. (12) for the generating function of the Legendre polynomials with respect to x from 1 to z, we obtain the generating function for the polynomials $\Phi_k(z)$:

$$\frac{1}{t} - 1 - \frac{\sqrt{1-2zt+t^2}}{t} = \sum_{k=0}^{\infty} \Phi_k(z) t^k, \qquad \Phi_k(z) = \int_1^z P_k(x)\,dx. \tag{17}$$

These polynomials can be presented via the polynomials $P_k(z)$ not only in the form of the integral in Eq. (17), but also in the form of the algebraic formula

$$\Phi_k(z) = [P_{k+1}(z) - P_{k-1}(z)]/(2k+1), \qquad k = 0, 1, 2, \dots\ , \tag{18}$$

which follows from the recurrent dependence [17, 18] relating the Legendre polynomials to their derivatives. In accordance with the properties of spherical functions, we assume that $P_{-1}(z) = P_1(z) = z$ in Eq. (18) [17]. Using recurrent formulas for the Legendre polynomials, we can obtain the corresponding formulas for the polynomials $\Phi_k(z)$, in particular,

$$\Phi_0(z) = z - 1, \qquad \Phi_1(z) = (z^2-1)/2, \qquad \Phi_2(z) = (z^3 - z)/2,$$

$$(k+2)\Phi_{k+1}(z) + (k-1)\Phi_{k-1}(z) = (2k+1)z\Phi_k(z), \qquad k = 1, 2, \dots\ .$$

The functions $\Phi_k(z)$ are introduced so that the equality $\Phi_k(1) = 0$ is satisfied.

Using the expressions obtained above, we can write Eq. (14) as follows:

$$Q_s(\tau) = 2\Big(\tau + \sqrt{\frac{\tau}{\pi}}\,\Big) + \tau \sum_{k=0}^{\infty} \frac{(-1)^k \Phi_k(\gamma)\tau^{k/2}}{\Gamma(k/2+2)}. \tag{19}$$

By virtue of the above-indicated properties of the polynomials $\Phi_k(z)$ [$\Phi_k(1) = 0$] at $\gamma = 1$, Eq. (19) transforms to Eq. (16). Similarly, the asymptotic expansion (15) can be written in a more compact form in terms of the polynomials $\Phi_k(z)$:

$$Q_s(\tau) = 2\Big(\tau + \sqrt{\frac{\tau}{\pi}}\,\Big) + \frac{1-\gamma^2}{2} - \frac{\sqrt{\tau}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k}\, \Phi_{2k}(\gamma)\Gamma(k-1/2), \qquad \tau \to \infty. \tag{20}$$

Examples of the Limiting Cases. The curves in Fig. 2 are qualitatively similar and close to each other in the ranges of the variables considered. The dependences $Q_s(\tau)$ corresponding to these curves depend only weakly on the parameter γ in the range $\gamma \in (0, 2]$. In addition, it should be noted that the solution is much simpler at $\gamma = 1$. Thus, it is reasonable to study the problem solution in the neighborhood of the point $\gamma = 1$ by the method of the small parameter. This means that the solution of the impregnation problem should be expanded with respect to the powers of the parameter $\gamma - 1$. Note that the expansions are finite sums because many of the formulas derived above contain polynomials [$P_k(z)$ or $\Phi_k(z)$]. The calculations yield the formulas

$$\Phi_k^{(n+1)}(1) = P_k^{(n)}(1) = \frac{1}{2^n n!}\,\frac{(k+n)!}{(k-n)!}, \qquad n \leqslant k,$$

where the superscript in brackets means the order of the derivative of the corresponding quantity with respect to the parameter γ. After some simple transformations, the solution of problem (19) can be presented as

$$Q_s(\tau) = 2\Big(\tau + \sqrt{\frac{\tau}{\pi}}\,\Big) + \tau(\gamma - 1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\frac{(\gamma-1)^k \tau^{k/2}}{2^k}\,\Omega_k(\tau),$$

where

$$\Omega_k(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{n/2}}{\Gamma(n/2 + k/2 + 2)}\,\frac{(2k+n)!}{n!}.$$

We confine ourselves to the first-order correction with respect to the parameter $\gamma - 1$. For this purpose, we sum up the series determining the function $\Omega_0(\tau)$. As a result, we obtain the relation

$$Q_s(\tau) = 2[\tau + \gamma(\tau/\pi)^{1/2}] + (1 - \gamma)[1 - \exp(\tau)\,\mathrm{erfc}\,(\tau^{1/2})], \tag{21}$$

where $\mathrm{erfc}\,(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int\limits_z^{\infty} \exp(-t^2)\,dt$ is an additional probability integral [$\mathrm{erfc}\,(\tau) = 1 - \mathrm{erf}\,(\tau)$].

Note that various presentations for the functions $\Omega_k(\tau)$ [in particular, for $\Omega_0(\tau)$] can be obtained by expanding the Laplace-transformed function $Q_s^*(p)$ (20) into a series with respect to the powers of $\gamma - 1$, with subsequent inversion of the expansion coefficients obtained (functions of the parameter p).

Calculations by Eq. (21) show that this dependence offers a satisfactory description of the exact solution plotted in Fig. 2 in the examined range $\gamma \in (0, 2]$.

Let us consider the limiting variant $\gamma \to 0$, which can be easily analyzed (as well as the case $\gamma \to 1$) on the basis of dependences obtained in the present work. In particular, it is rather easy to obtain a two-term expansion for the flux density $F_s(\tau)$ and for $Q_s(\tau)$:

$$\begin{gathered} F_s(\tau) = 1 + \frac{\exp(-\tau)}{\sqrt{\pi\tau}} + \mathrm{erf}\,(\sqrt{\tau}\,) + \gamma \exp\Big(\frac{-\tau}{2}\Big) I_0\Big(\frac{\tau}{2}\Big) + O(\gamma^2), \\ Q_s(\tau) = \tau + \sqrt{\frac{\tau}{\pi}}\,\exp(-\tau) + \Big(\frac{1}{2} + \tau\Big)\,\mathrm{erf}\,(\sqrt{\tau}\,) + 2\gamma \int\limits_0^{\tau/2} \exp(-\xi)\, I_0(\xi)\,d\xi + O(\gamma^2). \end{gathered} \tag{22}$$

Here $I_0(z)$ is the Bessel function of the imaginary argument [17, 18]. The corrections in Eq. (22) are uniformly applicable in the entire range $\tau \in (0, \infty)$, though the model itself is applicable only in a limited time interval, as was noted above.

Generalization of the Solution. With allowance for the results obtained, the solution of the global problem of determining the amount of the target component that entered the porous body under the boundary condition $G(0,\tau) = f(\tau)$ can be presented as the integral of convolution of the functions

$$Q_s(\tau) = \int\limits_0^{\tau} f(\tau - \xi) F_s(\xi)\,d\xi, \tag{23}$$

where $F_s(\tau)$ is the flux density under the unit [see Eq. (7)] boundary condition in the case with $z = 0$:

$$F_s(\tau) = 2 + \frac{1}{\sqrt{\pi\tau}} + \sum_{k=0}^{\infty} \frac{(-1)^k \Phi_k(\gamma)\tau^{k/2}}{\Gamma(k/2+1)}.$$

The quantity F_s is obtained by term-by-term differentiation of series (19). At small values of τ, the integral in Eq. (23) can be easily calculated numerically, for instance with the trapezium rule. As $\tau \to 0$, the singularity of the function $F_s(\tau)$ can be easily eliminated in calculating the integral in Eq. (23) by replacement of the integration variable $\xi = x^2$. An asymptotic (as $\tau \to \infty$) expansion of the function $F_s(\tau)$ can be useful in calculations. This expansion can be easily obtained by differentiating the asymptotic series (20) or by using dependence (10):

$$F_s(\tau) = 2 + \frac{1}{\sqrt{\pi\tau}} + \frac{1}{\pi\sqrt{\tau}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} \Phi_{2k}(\gamma)\Gamma(k+1/2), \qquad \tau \to \infty.$$

Distribution of the Substance over Large and Small Channels. It seems of interest to study the distribution of the target component over the channels of various types. The total amount of the target component in the large channel can be found by integrating Eq. (9):

$$Q_1^*(p) = \int_0^{\infty} \frac{\exp\left[z(1-\sqrt{p+2\gamma\sqrt{p}+1}\,)\right]}{p}\, dz = \frac{1+\sqrt{p+2\gamma\sqrt{p}+1}}{p(p+2\gamma\sqrt{p}\,)}.$$

The law of conservation of matter yields the amount of the target component in the small pores:

$$Q_2^*(p) = Q_s^*(p) - Q_1^*(p) = 2\gamma/[p^{3/2}(\sqrt{p+2\gamma\sqrt{p}+1}-1)].$$

Similar to dependences (14) and (15), we can derive the formulas for $Q_1(\tau)$ and $Q_2(\tau)$:

$$Q_1(\tau) = \tau \sum_{k=0}^{\infty} \frac{(-1)^k (2\gamma)^k}{\Gamma(2+k/2)}\, \tau^{k/2} + \sqrt{\tau} \sum_{k=0}^{\infty} \frac{(-1)^k P_k(\gamma)}{\Gamma(k/2+3/2)}\, \tau^{k/2} + \tau^{3/2} \sum_{k=0}^{\infty} \frac{(-1)^k R_k(\gamma)}{\Gamma(k/2+5/2)}\, \tau^{k/2},$$

$$R_n(x) = \sum_{k=0}^{n} P_k(x)(2x)^{k-n},$$

$$Q_1(\tau) \simeq 1 - \frac{1}{(2\gamma)^2} - \frac{1}{\pi\sqrt{\tau}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} P_{2k+1}(\gamma)\Gamma(k+1/2) - \frac{\sqrt{\tau}}{2\gamma\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} \frac{\Gamma(k-1/2)}{(2\gamma)^{2k}}$$

$$-\frac{\sqrt{\tau}}{2\gamma}\Big(\gamma + \frac{1}{2\gamma} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} S_{2k}(\gamma)\Gamma(k-1/2)\Big), \quad S_n(x) = \sum_{k=0}^{n} \frac{P_k(x)}{(2x)^{k-n}}, \quad \tau \to \infty,$$

$$Q_2(\tau) = 2\gamma\tau\Big(\sqrt{\tau} \sum_{k=0}^{\infty} \frac{(-1)^k (2\gamma)^k}{\Gamma(5/2+k/2)}\, \tau^{k/2} + \sum_{k=0}^{\infty} \frac{(-1)^k P_k(\gamma)}{\Gamma(k/2+2)}\, \tau^{k/2} + \tau \sum_{k=0}^{\infty} \frac{(-1)^k R_k(\gamma)}{\Gamma(k/2+3)}\, \tau^{k/2}\Big),$$

$$Q_2(\tau) \simeq 2\tau - \frac{\gamma^2}{2} + \frac{1}{2\gamma^2} - \frac{2\gamma\sqrt{\tau}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} P_{2k}(\gamma)\Gamma(k-1/2) + \frac{\sqrt{\tau}}{2\gamma\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} \frac{\Gamma(k-1/2)}{(2\gamma)^{2k}}$$

$$+\frac{\sqrt{\tau}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\tau^k} S_{2k+1}(\gamma)\Gamma(k-1/2), \quad \tau \to \infty.$$

In practice, where the motion of the liquid phase in the porous body is initiated by a short pressure pulse on the boundary (for instance, during implosion of a gas bubble), the proposed model with $U \neq 0$ should be used in a limited time interval $t < T_5$ (T_5 is the pulse duration). After that, it is necessary to pass to a simpler equation (5) with $U = 0$. In this case, there is a non-zero distribution of the initial concentration of the target component over the pore volume at the beginning of the process. In the case of interest for practice, apparently, no significant amount of the target component is still accumulated in the small pores after the pressure pulse, whereas the amount of the target component in the transportation channel should be taken into account. In this case, the initial stage of impregnation described by model (6), (7) is usually considered, i.e., Eq. (14) or (19).

Finally, we consider the case with $D_1 \to 0$ in Eq. (5). Here, it is reasonable to use $T_3 = b^{-2}$ and $l_3 = U/b^2$ as the time and length scales. Introducing the Peclet number $\mathrm{Pe} = Ul_3/D_1 = U^2/(D_1 b^2)$, we can easily see that the case with $D_1 \to 0$ corresponds to the limiting variant $\mathrm{Pe} \to \infty$ and with the time and length scales $T_4/T_3 = O(\gamma^2)$ and $l_4/l_3 = O(\gamma^2)$ ($\gamma = bD_1^{1/2}/U \to 0$), which are larger than those used in Eq. (6).

In the dimensionless coordinates $\zeta = t/T_3$ and $\xi = x/l_3$, Eq. (5) with $D_1 = 0$ is written as

$$\Big(\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\xi} + \frac{\partial^{1/2}}{\partial\zeta^{1/2}}\Big) G = 0, \qquad G = G(\xi, \zeta), \tag{24}$$

and under additional conditions, it becomes

$$G(0, \zeta) = 1, \qquad G(\xi, 0) = 0. \tag{25}$$

In the problem considered, the length of the domain along the X axis is not necessarily infinite, i.e., l_1 can be considered as a finite quantity.

The solution of problem (24), (25) is easily found by the operational method and has the form

$$G(\xi, \zeta) = \operatorname{erfc}\Big(\frac{\xi}{2\sqrt{\zeta - \xi}}\Big) H(\zeta - \xi),$$

where $H(z)$ is the Heaviside function [$H(z) = 1$ at $z > 0$ and $H(z) = 0$ at $z < 0$]. It should be noted that the solution obtained has a wavy character. The results of solving this problem can be used to solve the problem of removing admixtures from a porous body during its washing [for instance, with replacement of the sought functions described by Eq. (1)].

Conclusions. An equation (5) is proposed to describe the initial stage of impregnation of a porous system. This equation yields analytical expressions for determining the amount of the substance penetrating into the body: dependences (14), (19) and (15), (20), applicable for calculations at short and long times, respectively. One possible limiting variant of the model proposed in the present work is considered.

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